LINEARIZED PROBLEM OF SUPERSONIC FLOW AT THE ENTRANCE TO THE ELECTRODE REGION OF A MAGNETOHYDRODYNAMIC CHANNEL

(LINEARIZOVANNAIA ZADACHA O SVERKHZVUKOVOM TECHENII NA VKHODE V ELEKTRODNUIU ZONU MAGNITOGIDRODINAMICHESKOGO KANALA)

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It is usually assumed that in magnetohydrodynamic installations the length of the channel is much greater than its width. This permits the performance of flow calculations by the one-dimensional scheme. However, these calculations are not applicable to regions where a sharp change in parameters of the applied electromagnetic field takes place (over a length of the order of channel width). Clarification of the character of the flow in such regions is therefore of interest. In [1 and 2] electrical fields and fields of currents in the region of entrance of flow into the magnetic field and in the region between electrodes were studied. In these cases the flow was considered as given and the influence of electromagnetic field on flow was neglected. In [3] the linearized problem of the influence of electromagnetic field on the flow of an incompressible fluid in the vicinity of the electrode ends is examined. The magnetic field was assumed to be constant and different from zero in the region between the electrodes.

In this paper the influence of the electromagnetic field on supersonic gas flow is examined. Since the interaction parameter and the magnetic Reynolds number are usually small, the problem was examined in the linearized formulation. The magnetic field was considered as specified and variable along the length of the channel.

Formulation of problem. We shall examine the steady two-dimensional problem of supersonic flow of a conducting gas in a flat channel $-a < y < a$, $-\infty < x < \infty$. The walls of the channel for $x < 0$ are insulators, for $x > 0$ conductors. The gas is assumed to be perfect and ideal with constant electrical conductivity σ. Ohm's law is taken in the form

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{H}\right)$$

which is valid for a sufficiently high gas density.

For $x = -\infty$ the gas moves with translational supersonic velocity U while the electric field E is equal to zero. In finding the electrical field it is assumed that the field is bounded at $x = \infty$.

Assume the magnetic Reynolds number R_m and the interaction parameter N to be small

$$R_m = \frac{4\pi\sigma U a}{c^2}, \qquad N = \frac{\sigma H_0^2 a}{\rho U c^2}$$

The smallness of R_m permits the induced magnetic field to be neglected.

Let the given field have the form

$$\mathbf{H} = H(x)\,\mathbf{e}_z, \qquad H(x) = \begin{cases} H_0 & \text{for } x \geqslant 0 \\ H_0 (k^2+1)\, e^{\pi x/a} (1 + k^2 e^{\pi x/a})^{-1} & \text{for } x < 0 \end{cases} \tag{1}$$

Here $\mathbf{e}_z$ is the unit vector perpendicular to the plane of flow, k is a parameter characterizing the profile of the magnetic field.

Since the interaction parameter N is assumed to be small, the perturbations of velocity and thermodynamic quantities are also small.

In the first approximation, therefore, the electric field and currents are determined by the given constant velocity of the gas and by the magnetic field. Then the electromagnetic force and the Joule heat are computed, and from linearized equations of motion all hydrodynamic parameters are found. In this fashion the general system of magnetohydrodynamic equations in this case is split up into two systems with corresponding boundary conditions.

For determination of electric field $\mathbf{E} = \operatorname{grad} \varphi$ and the current we have equations of continuity of current and Ohm's law, i.e.

$$\operatorname{div} \mathbf{j}' = 0, \qquad \mathbf{j}' = [(\operatorname{grad} \varphi)' - h(x)\,\mathbf{e}_y] \tag{2}$$

Here primes designate nondimensional quantities determined by Equations

$$x' = \frac{x}{a}, \quad y' = \frac{y}{a}, \quad \mathbf{j}' = \frac{4\pi a}{HcR_m}\mathbf{j}, \quad h(x) = \frac{H(x)}{H_0}, \quad (\operatorname{grad}\varphi)' = \frac{\operatorname{grad}\varphi}{H_0 U / c}$$

Since the potential of electrodes is constant and since on insulators the normal component of current is absent, we have the following boundary conditions for the nondimensional potential (3)

$$\varphi' = \pm\eta \equiv \frac{\varphi_0}{aH_0 U / c} \quad \text{for } y' = \pm 1,\ x' > 0, \quad \frac{\partial \varphi'}{\partial y'} = h(x) \quad \text{for } y' = \pm 1,\ x' < 0$$

$$|(\operatorname{grad}\varphi)'| = 0 \quad \text{for } x' = -\infty, \qquad (\operatorname{grad}\varphi)' = \eta\mathbf{e}_y \quad \text{for } x' = \infty$$

The last condition arises from the boundedness of the electric field at $x = \infty$. Linearized hydrodynamic equations have the form

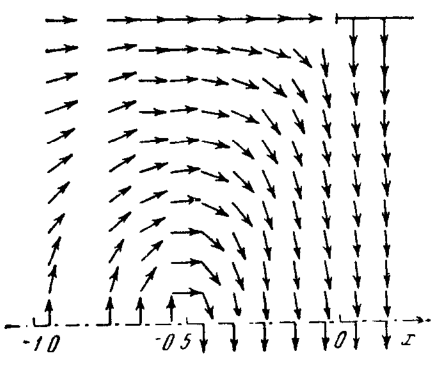

Fig. 1

$$\frac{\partial u'}{\partial x'} + \frac{1}{M^2}\frac{\partial \rho'}{\partial x'} + \frac{1}{\gamma M^2}\frac{\partial \xi'}{\partial x'} = F_x \tag{4}$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial \rho'}{\partial x'} = 0$$

$$\frac{\partial v'}{\partial x'} + \frac{1}{M^2}\frac{\partial \rho'}{\partial y'} + \frac{1}{\gamma M^2}\frac{\partial \xi'}{\partial y'} = F_y$$

$$\frac{\partial \xi'}{\partial x'} = \gamma(\gamma - 1) M^2 Q$$

$$F_x = h(x)\left[\frac{\partial \varphi'}{\partial x'} - h(x)\right]$$

$$F_y = -\frac{\partial \varphi'}{\partial x'} h(x)$$

$$Q = [(\operatorname{grad}\varphi)' - h(x)\,\mathbf{e}_y]^2$$

Nondimensional quantities are introduced by Equations

$$\mathbf{v}' = u'\mathbf{e}_x + v'\mathbf{e}_y = \frac{\mathbf{v}}{U}\frac{1}{N}, \qquad \rho' = \frac{\rho}{\rho_0}\frac{1}{N}, \qquad \xi' = \frac{s}{Nc_v}$$

Here ρ_0 is the value of the unperturbed density, ρ , $\mathbf{v}$ and s are perturbations of density, velocity and entropy, respectively.

In this connection the perturbation of velocity must satisfy the boundary conditions

$$v' = 0 \quad \text{for } y' = \pm 1, \qquad \mathbf{v}' = 0 \quad \text{for } x' = -\infty$$

As is known, system (2) is reduced to Laplace's equation for φ with mixed boundary conditions (3). This problem can be solved [1 and 2] through the equation of Keldysh-Sedov. For a magnetic field given by equality (1) the solution has the form

$$\begin{aligned}\frac{\partial\varphi'}{\partial y'} = &-\frac{(k^2+1)(e^{2\alpha}\cos 2\beta - k^2e^{4\alpha})}{1+k^4e^{4\alpha}-2k^2e^{2\alpha}\cos 2\beta} + \\ &+ A(\alpha,\beta)\frac{\sqrt{k^2+1}}{k^2}\,\frac{1-k^2e^{\alpha}\sin\beta - k^2e^{2\alpha}\cos 2\beta - k^4e^{3\alpha}\sin\beta}{1-2k^2e^{2\alpha}\cos 2\beta + k^4e^{4\alpha}} - \\ &- B(\alpha,\beta)\sqrt{k^2+1}\,\frac{e^{\alpha}\cos\beta + e^{2\alpha}\sin 2\beta - k^2e^{3\alpha}\cos\beta}{1-2k^2e^{2\alpha}\cos 2\beta + k^4e^{4\alpha}} + \\ &+ A(\alpha,\beta)\left(\eta - \frac{k^2+1}{k^2}\right) + \gamma_0\,\mathrm{sign}\,\beta\left(\frac{\sqrt{1+2e^{2\alpha}\cos 2\beta + e^{4\alpha}} - 1 - e^{2\alpha}\cos 2\beta}{2+4e^{2\alpha}\cos 2\beta + 2e^{4\alpha}}\right)^{1/2}\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{\partial\varphi'}{\partial x'} = &-\frac{(k^2+1)e^{2\alpha}\sin 2\beta}{1+k^4e^{4\alpha}-2k^2e^{2\alpha}\cos 2\beta} + \\ &+ A(\alpha,\beta)\sqrt{k^2+1}\,\frac{e^{\alpha}\cos\beta + e^{2\alpha}\sin 2\beta - k^2e^{3\alpha}\cos\beta}{1-2k^2e^{2\alpha}\cos 2\beta + k^4e^{4\alpha}} + \\ &+ B(\alpha,\beta)\frac{\sqrt{k^2+1}}{k^2}\,\frac{1-k^2e^{\alpha}\sin\beta - k^2e^{2\alpha}\cos 2\beta - k^4e^{3\alpha}\sin\beta}{1-2k^2e^{2\alpha}\cos 2\beta + k^4e^{4\alpha}} + \\ &+ B(\alpha,\beta)\left(\eta - \frac{k^2+1}{k^2}\right) - \gamma_0\left(\frac{\sqrt{1+2e^{2\alpha}\cos 2\beta + e^{4\alpha}} + 1 + e^{2\alpha}\cos 2\beta}{2+4e^{2\alpha}\cos 2\beta + 2e^{4\alpha}}\right)^{1/2}\end{aligned}$$

$$A(\alpha,\beta) = \left(\frac{\sqrt{1+2e^{2\alpha}\cos 2\beta + e^{4\alpha}} - 1 + e^{2\alpha}}{2+4e^{\alpha}\sin\beta + 2e^{2\alpha}}\right)^{1/2} \qquad \left(\alpha = \frac{\pi x}{2a},\ \beta = \frac{\pi y}{2a}\right)$$

$$B(\alpha,\beta) = \left(\frac{\sqrt{1+2e^{2\alpha}\cos 2\beta + e^{4\alpha}} + 1 - e^{2\alpha}}{2+4e^{\alpha}\sin\beta + 2e^{2\alpha}}\right)^{1/2} \quad \left(\gamma_0 = -\eta - \frac{\sqrt{k^2+1}}{k^2} + \frac{k^2+1}{k^2}\right)$$

We note that the electric field in the channel is independent of the form of the magnetic field in the region between the electrodes since the field enters only through the boundary conditions on the insulator.

Using these expressions we can calculate from (2) and (4) the distribution of currents, of forces and of dissipation in the channel. Figs. 1 and 2 present the field of directions of electric current and the distribution of dissipation across the channel for different values of x at $k = 1$.

From system (4) we eliminate ρ and ξ and obtain

$$\frac{\partial u'}{\partial x'}(M^2-1) - \frac{\partial v'}{\partial y'} = M^2\,[F_x - (\gamma-1)\,Q] \qquad -\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} = F_y - \int_{-\infty}^{x'}\frac{\partial F_x}{\partial y}\,dx' \tag{6}$$

In this connection ρ' and ξ' are found from Equations

$$\rho' = -M^2 u' - \frac{1}{\gamma}\xi' + M^2 \int_{-\infty}^{x'} F_x\,dx', \qquad \xi' = \gamma(\gamma-1)\int_{-\infty}^{x'} Q\,dx' \tag{7}$$

Equations of characteristics of the system of equations (6) have the form

$$y' = \pm\frac{1}{\sqrt{M^2-1}}x', \qquad \frac{\partial}{\partial x'}\left[u' \mp \frac{1}{\sqrt{M^2-1}}v'\right] = \tag{8}$$

$$= -\frac{M^2}{\sqrt{M^2-1}}\left\{(\gamma-1)Q - F_x \pm \frac{M^2}{\sqrt{M^2-1}}\left(F_y - \frac{\partial}{\partial y'}\int_{-\infty}^{x'} F_x\,dx'\right)\right\}$$

For numerical solution of the problem an orthogonal net of points x'_n and y'_m was selected in such a manner that $\Delta y' \equiv y'_{m+1} - y'_m = 0.1$, $\Delta x'$ was selected equal to $\sqrt{M^2-1}\,\Delta y'$.

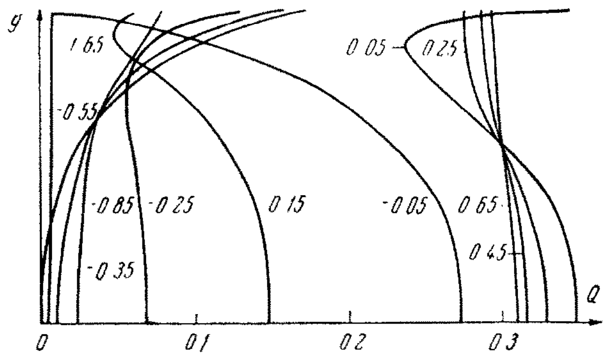

Fig. 2

For determination of values u', v', ρ' and ξ' at the point $x'_n y'_m$ the values of these quantities at points

$$x'_{n-1}y'_{m-1},\ x'_{n-1}y'_m,\ x'_{n-1}y'_{m+1},$$

were utilized. For computation of integrals in solving Equations (8) and (7), the trapezoid equation without interstitial points between points of the net was utilized here.

Computations were carried out on the "Strela" electronic computer for combinations of parameters presented in Table 1.

Table 1

k	M	η	γ
1	1.41	0.45	1.4
1	1.41	0.5	1.4
1	1.41	0.3	1.4
1	2.0	0.45	1.4
0.1	1.41	0.45	1.4

Table 2

k	X°	$X^\vee$	η
0.1	—0.35	—0.05	0.45
1	—0.55	—0.05	0.45
10	—1.55	—0.55	0.3
1	—0.65	—0.15	0.3

Results. The presence of ring currents with a singular point like a center on the axis of the channel at the entrance of the flow into the magnetic field (Fig.1) appears as a characteristic peculiarity of the electric current field. The region of ring currents is separated from the region where the currents are closed through the electrodes by a separatrix which terminates on the walls of the channel at $x < 0$ in saddle singular points.

On the electrode ends the current density becomes infinite, however, the total current, the total amount of Joule heat and the impulse which are transferred to the fluid in any region near these singular points are finite.

With increase in parameter k, i.e. with motion of the magnetic field out beyond the electrodes, the region of ring currents also moves away upstream. Thus, values of abscissas of the singular point center X° and of the saddle singular point $X^{\vee}$ are presented in Table 2.

If the load coefficient is decreased, the region of ring currents is also displaced upstream and the density of currents increases.

In the downstream direction the field of currents rapidly becomes one-dimensional. This, at one quarter of channel width from the electrode ends ($x' \sim 0.5$) the ratio of currents $j_x/j_y \sim 0.015$ ($k = 1$).

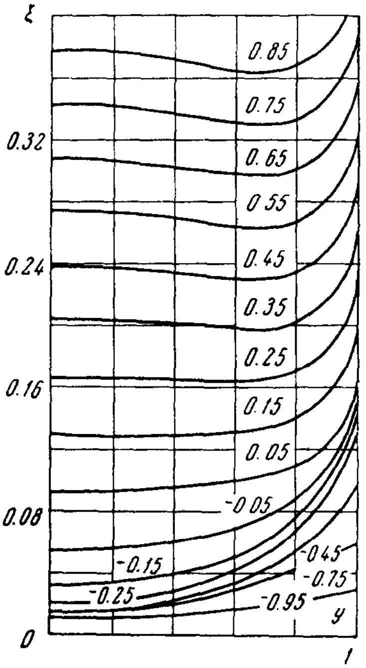

Fig. 3

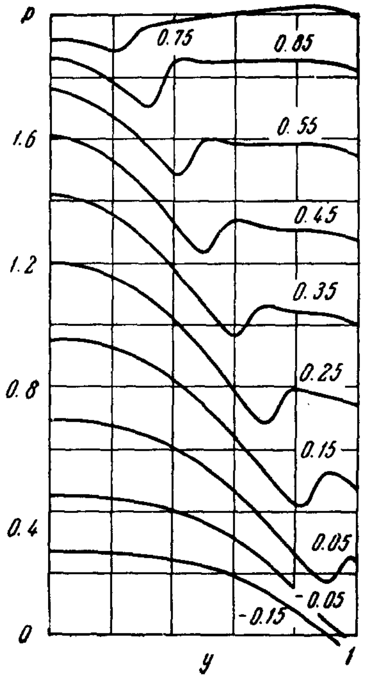

Fig. 4

Joule dissipation is small for large negative x' (Fig.2). (In Fig.2 and in the following figures the values of x' for which the curves are presented are shown next to the corresponding curves). For example $Q < 0.05$ for $k = 1$ at $x' = -1.65$.

For x' comparable to $X^{\vee}$, dissipation at the wall increases at the expense of increase in current density. In the vicinity of the singular point center the dissipation decreases. In the vicinity of the separation line between currents (separatrix) dissipation in the central part of the channel increases sharply. At the walls it is the opposite; dissipation decreases to zero in the vicinity of the saddle singular point. In the vicinity where the electrode starts, a sharp increase in dissipation occurs near the walls. After that a smoothing of dissipation over the cross-section takes place. At $x' = 0.85$ for $k = 1$ and $\eta = 0.45$ the relative change of Q across the channel is of the order of 0.03.

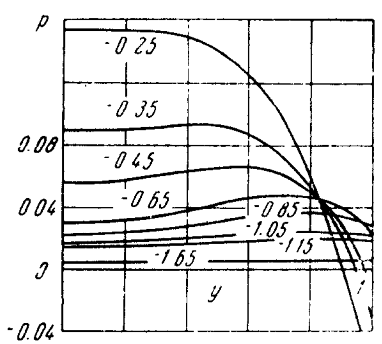

Fig. 5

Since for decrease in k the saddle singular point approaches $x' = 0$, the total quantity of heat, given off in the vicinity where the electrode starts, decreases due to small dissipation in the vicinity of the saddle singular point.

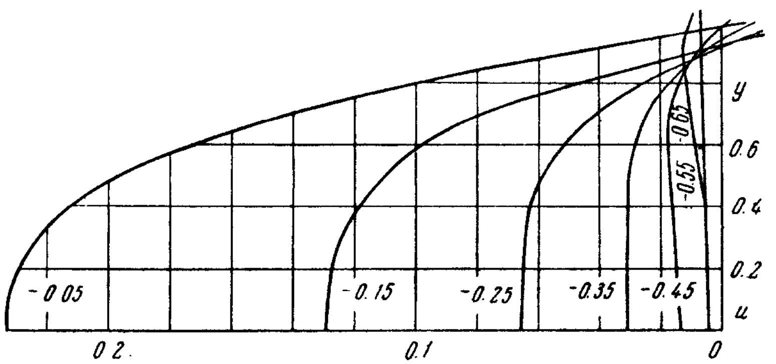

Fig. 6

With decreases in the load coefficient dissipation Q increases.

The values of Q on the axis of the channel for $k = 1$ are presented for three values of x'

$x' = -0.25$	-0.05	0.65	
$Q = 0.06$	0.27	0.31	for $\eta = 0.45$
$Q = 0.12$	0.39	0.49	for $\eta = 0.30$

Dissipation determines the increase in entropy. A plot of entropy for $k = 1$, $M = 1.4$, $\eta = 0.45$ is presented in Fig.3.

For large negative x' the pressure changes little across the channel. Thus, for $k = 1$, $M = 1.4$ and $\eta = 0.45$ at $x' = -1.15$ the relative change of pressure across the channel is of the order of 0.2 . For x' of the order of $X^{\vee}$, near the wall a transverse force appears directed towards the axis of the channel. This force decreases the pressure in the indicated region. As a result of this a region of rarefaction arises on some part of the wall. In this region a flattened expansion wave is formed.

In the vicinity where the electrodes start, a compression wave is formed at the expense of increase in dissipation and decelerating force. With decrease in k this compression wave becomes weaker due to decrease in heat discharge in this region.

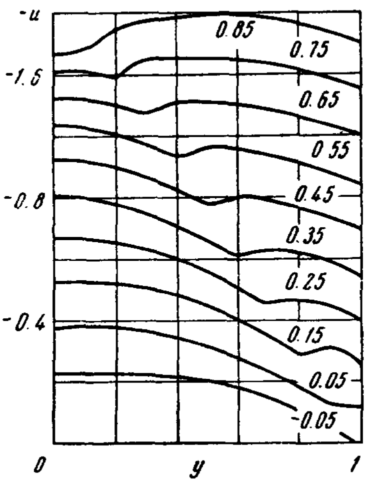

Fig. 7

After transition through the region of strong dissipation an expansion wave is formed which is particularly intensely manifested in the case $k = 1$, $\eta = 0.3$ and $M = 1.4$ when the dissipation in the vicinity where the electrodes begin is especially great.

For positive values of $x'(x' > 0.25)$ a linear increase with respect to x' of the average pressure over the cross-section occurs. Waves mentioned above propagate against this background. Curves for pressure distribution are presented in Figs. 4 and 5.

For large negative x' uniform deceleration of the stream takes place ($u < 0$) as a result of heat discharge due to dissipation (Figs. 6 and 7).

Further downstream the deceleration in the center part decreases due to increasing accelerating force. For x' close to $\sim$ acceleration of the stream near the wall takes place produced by the negative pressure gradient. In the vicinity of the separatrix sharp deceleration of the stream commences. Further a linear, with respect to x', decrease of the average magnitude of velocity over the cross section occurs.

The magnitude of perturbation of transverse velocity is by an order smaller than the magnitude of perturbation of longitudinal velocity, i.e. $v'/u' \sim 0.1$. In the region of ring currents a compression of the stream takes place towards the center ($v' < 0$ for $y > 0$) due to transverse forces. After passing through the center the stream expands again and a compression wave propagates through the stream from the electrode ends. Further these waves propagate downstream.

In conclusion we note that approximately one quarter of channel width downstream from the beginning of electrodes the stream already becomes practically uniform and from here down can be computed from one-dimensional theory. As initial data for such a calculation it is necessary to take quantities which are obtained from two-dimensional theory after their averaging over the width of the channel. If one is interested not only in the average characteristics of the stream, it is possible to examine the propagation of waves, which arise at the entrance and which in this paper are computed by the linear theory, against a background which is computed by one-dimensional nonlinear theory.

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